FAST DISCOVERING THE FUTURE

Mathematics for Machine Learning

Lab 3

Problem 1. Show that

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1 v_1 - 2(u_2 v_1 + u_1 v_2) + 5 u_2 v_2$$

is an inner product for vectors $\boldsymbol{u} = [u_1, u_2]^T$ and $\boldsymbol{v} = [v_1, v_2]^T$. Find a matrix $A \in \mathbb{R}^{2\times 2}$ such that

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^T A \boldsymbol{v}.$$

Problem 2. Check whether these matrices are positive definite.

1) $\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$ **2)** $\begin{bmatrix} 1 & -3 \\ -3 & 8 \end{bmatrix}$ **3)** $\begin{bmatrix} 5 & 1 \\ -1 & 5 \end{bmatrix}$

Problem 3. Compute $\langle \boldsymbol{u}, \boldsymbol{v} \rangle, ||\boldsymbol{u}||, ||\boldsymbol{v}||$ and $d(\boldsymbol{u}, \boldsymbol{v})$ first using the inner product $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^T \boldsymbol{v}$ and then the inner product from the first exercise for the given vectors. What is the angle between the vectors w.r.t these inner products?

1) $\boldsymbol{u} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ **3)** $\boldsymbol{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2) $\boldsymbol{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ **4)** $\boldsymbol{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Problem 4. Check whether the following matrices (A) are orthogonal and if so, compute the norm of $A\boldsymbol{x}$ for

1) $A = \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ -6/7 & 3/7 & 2/7 \\ 2/7 & 6/7 & -3/7 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$

2) $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} 0 \\ 2 \\ -7 \end{bmatrix}$

3) $A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}, \boldsymbol{x} = \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}$

Problem 5. Show that the vectors

$$b_1 = \begin{bmatrix} 1/2 \\ 0 \\ -\sqrt{3}/2 \end{bmatrix} \qquad b_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad b_3 = \begin{bmatrix} \sqrt{3}/2 \\ 0 \\ 1/2 \end{bmatrix}$$

form an orthonormal basis for $\mathbb{R}^3$.

Problem 6. Compute the projection matrix P_π onto the line through the origin spanned by $\boldsymbol{b} = [1 \quad -1 \quad 2]^T$. Project the given vector $\boldsymbol{x} = [1 \quad 2 \quad 3]^T$ onto the subspace U spanned by $\boldsymbol{b}$.

Problem 7. Find the eigenvalues and their corresponding eigenspaces for the given matrices

1) $\begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix}$ **2)** $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ **3)** $\begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}$