Mathematics for Machine Learning: Homework 3
Deadline is 06.08.2020

July 29, 2020

1. Check that $\langle \cdot, \cdot \rangle$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle := 17x_1y_1 - 4(x_2y_1 + x_1y_2) + x_2y_2$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ is an inner product. Moreover, find a matrix $A \in \mathbb{R}^{2\times 2}$ so that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A\mathbf{y}.$$

2. Which of the following matrices is positive definite?

$$A = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 6 & -3 \\ -3 & 1 \end{bmatrix}.$$

3. Compute the norm of the vector $[3\ 2\ 1]^T$ w.r.t. the following inner products:

   a) $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T\mathbf{y}$,

   b) $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \begin{bmatrix} 3 & -2 & 0 \\ -2 & 4 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{y}$.

4. Find the angle between $\mathbf{x} = [2,\ 3]^T$ and $\mathbf{y} = [-3, 2]^T$ w.r.t. the inner products given by

   a) $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T\mathbf{y}$

   b) $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{y}$

5. Check that the following matrix (the rotation matrix) is orthogonal for any φ

$$A = \begin{bmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{bmatrix}.$$

Using orthogonality of the matrix A compute the Euclidean norm $\|A\mathbf{x}\|$ for $\mathbf{x} = [-4,\ 3,\ 12]^T$.

6. Check that the vectors

$$\mathbf{b}_1 = \frac{1}{13}\begin{bmatrix} 5 \\ 12 \end{bmatrix}, \quad \mathbf{b}_2 = \frac{1}{13}\begin{bmatrix} 12 \\ -5 \end{bmatrix}$$

form an orthonormal basis for $\mathbb{R}^2$.

7. Find the projection matrix P_π onto the line through the origin spanned by $\mathbf{b} = [1,\ 2,\ -4]^T$. Find also the projection of the vector $\mathbf{x} = [2,\ 3,\ 1]^T$.

8. Find the eigenvalues and eigenspaces of the matrix $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.