Mathematics for Machine Learning

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August 11, 2020

Limit of a Function

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Definition

Let $f : X \to \mathbb{R}$, $X \subset \mathbb{R}$ *is an interval and* $a \in X$*. It is said the limit of* f*, as* x *approaches* a*, is A and written* $\lim_{x \to a} f(x) = A$ *or* $f(x) \xrightarrow[x \to a]{} A$*, if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that from* $0 < |x - a| < \delta$, $x \in X$ *follows that* $|f(x) - A| < \varepsilon$*.*

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Theorem

Let $f \in C\left[a, b\right]$*. If* $f\left(a\right) f\left(b\right) < 0$ *then there exists* $c \in \left(a, b\right)$ *such that* $f\left(c\right) = 0$*.*

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If $f \in C[a,b]$, *then* f *has maximum and minimum values.*

Uniform Continuity

Definition

Let $f : X \to \mathbb{R}$*,* $X \subset \mathbb{R}$*. It is said* f *is uniformly continuous on the set* X*, if for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that from* $|x_1 - x_2| < \delta$*,* $x_1, x_2 \in X$ *follows that* $|f(x_1) - f(x_2)| < \varepsilon$*.*

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If $f \in C[a, b]$, *then it is uniformly continuous on* $[a, b]$.

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Let $f : X \to \mathbb{R}$, $X \subset \mathbb{R}$. *It is said* f *is differentiable at interior point* $x_0 \in X$, *if the following limit exists*

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Proposition

$$f(x_0 + h) - f(x_0) = f'(x_0)h + o(h), h \to 0.$$

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If f has a finite derivative at x_0 then it is continuous at x_0.