Mathematics for Machine Learning

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FAST DISCOVERING THE FUTURE

Series

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Definition

The series $\sum_{n=1}^{\infty} a_n$ is called convergent if A is finite, otherwise it is called divergent.

Series

Theorem

If $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $a_n \to 0$*. The inverse is not true.*

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Theorem

The series $\sum_{n=1}^{\infty} a_n$ *is convergent if and only if for every* $\varepsilon > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that for every natural* $n \geq n_0$ *and* m *holds*

$$|a_{n+1} + \ldots + a_{n+m}| < \varepsilon.$$

Theorem

If $a_n \geq 0$ *for all* $n \in \mathbb{N}$ *then* $\sum_{n=1}^{\infty} a_n$ *is convergent or it is equal to* $+\infty$.

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If $a_n, b_n \geq 0$ *and* $a_n \leq b_n$ *for all* $n \in \mathbb{N}$ *then if* $\sum_{n=1}^{\infty} b_n$ *is convergent, then* $\sum_{n=1}^{\infty} a_n$ *is convergent too.*

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Theorem

If $a_n, b_n \geq 0$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} \frac{a_n}{b_n} = K$, $0 \leq K < \infty$ *then if* $\sum_{n=1}^{\infty} b_n$ *is convergent, then* $\sum_{n=1}^{\infty} a_n$ *is convergent too.*

Series

Theorem

Let $a_n \geq 0$ *for all* $n \in \mathbb{N}$ *and* $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} = K$. *Then*

1. *if* $K < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *is convergent,*
2. *if* $K > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

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Theorem

Let $a_n \geq 0$ *for all* $n \in \mathbb{N}$. *Then*

1. *if* $\overline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n} < 1$ *then* $\sum_{n=1}^{\infty} a_n$ *is convergent,*
2. *if* $\underline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n} > 1$ *then* $\sum_{n=1}^{\infty} a_n$ *is divergent.*

Indefinite Integral

Definition

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Definition

The set of all antiderivatives of function f *is called indefinite integral of* f:

$$\int f(x)\,dx = F(x) + C.$$

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4. If $f, g \in C^1(X)$, then
$$\int f dg = fg - \int g df.$$

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Definition

Let $f : [a, b] \to \mathbb{R}$ *and* $\xi_i \in [x_i, x_{i+1}]$, $i = 0, \ldots, n-1$. *The sum* $\sigma = \sum\limits_{i=0}^{n-1} f(\xi_i) \Delta x_i$ *is called Riemann integral sum.*

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We will say that σ *tends to* I, *when* $\lambda \to 0$ *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any partition with diameter satisfying to* $\lambda < \delta$ *and for every Riemann inegral sum* σ *corresponding to that partitions holds* $|\sigma - I| < \varepsilon$.

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Theorem

If $f \in \mathcal{R}[a,b]$, then it is bounded. The inverse is not true.

Example

Dirichlet function is bounded but not integrable

$$D(x) = \begin{cases} 0, x \in \mathbb{Q} \cap [0,1], \\ 1, x \notin \mathbb{Q} \cap [0,1]. \end{cases}$$

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Taking $\xi_i \in \mathbb{Q}$ for all i, we will have $\sigma = 0$ and in the case of $\xi_i \notin \mathbb{Q}$ for all i we will have $\sigma = 1$.

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Remark

If X is not bounded from below (above), then we will denote $\inf X = -\infty$ $(\sup X = +\infty)$.

Definite Integral

Theorem

If $f : [a, b] \to \mathbb{R}$ is a bounded function, then it is integrable if and only if

$$\lim_{\lambda \to 0} \sum_{i=0}^{n-1} \omega_i \Delta x_i = 0,$$

where $\omega_i = M_i - m_i$, $M_i = \sup_{x \in [x_i, x_{i+1}]} f(x)$, $m_i = \inf_{x \in [x_i, x_{i+1}]} f(x)$.

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