Mathematics for Machine Learning

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FAST DISCOVERING THE FUTURE

Differential

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f is called di erentiable at the point (x_0, y_0) if there exists A, B 2 R such that

$$f(x_0 + \quad x, y_0 + \quad y) = f(x_0, y_0) + A \quad x + B \quad y + o(\rho), \rho \; ! \; 0,$$

where $\rho = \sqrt{\quad x^2 + \quad y^2}$.

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Theorem

If partial derivatives of the rst degree of f are continuous at (x_0, y_0) then it is di erentiable at (x_0, y_0). The inverse is not true.

Differential

Example

$$f(x,y) = \begin{cases} x^2 \sin \frac{1}{x}, \ \textit{if} \ (x,y) \neq (0,0), \\ 0, \ \textit{if} \ (x,y) = (0,0) \end{cases}$$

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Definition

$$df(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0) \quad x + \frac{\partial f}{\partial y}(x_0,y_0) \quad y \text{ is called differential of } f.$$

Directional Derivative

Theorem

Let $f : X \; ! \; \mathsf{R}$, $X \quad \mathsf{R}^n$ *and* x_0 *is an interior point of* X. *If* f *is di erentiable at* x_0, *then for all* v *2* R^n *such that* $kvk = 1$ *we have*

$$\lim_{t \; ! \; 0} \frac{f(x_0 + tv) \quad f(x_0)}{t} = \frac{\partial f}{\partial x_1}(x_0)\, v_1 + \ldots + \frac{\partial f}{\partial x_n}(x_0)\, v_n$$

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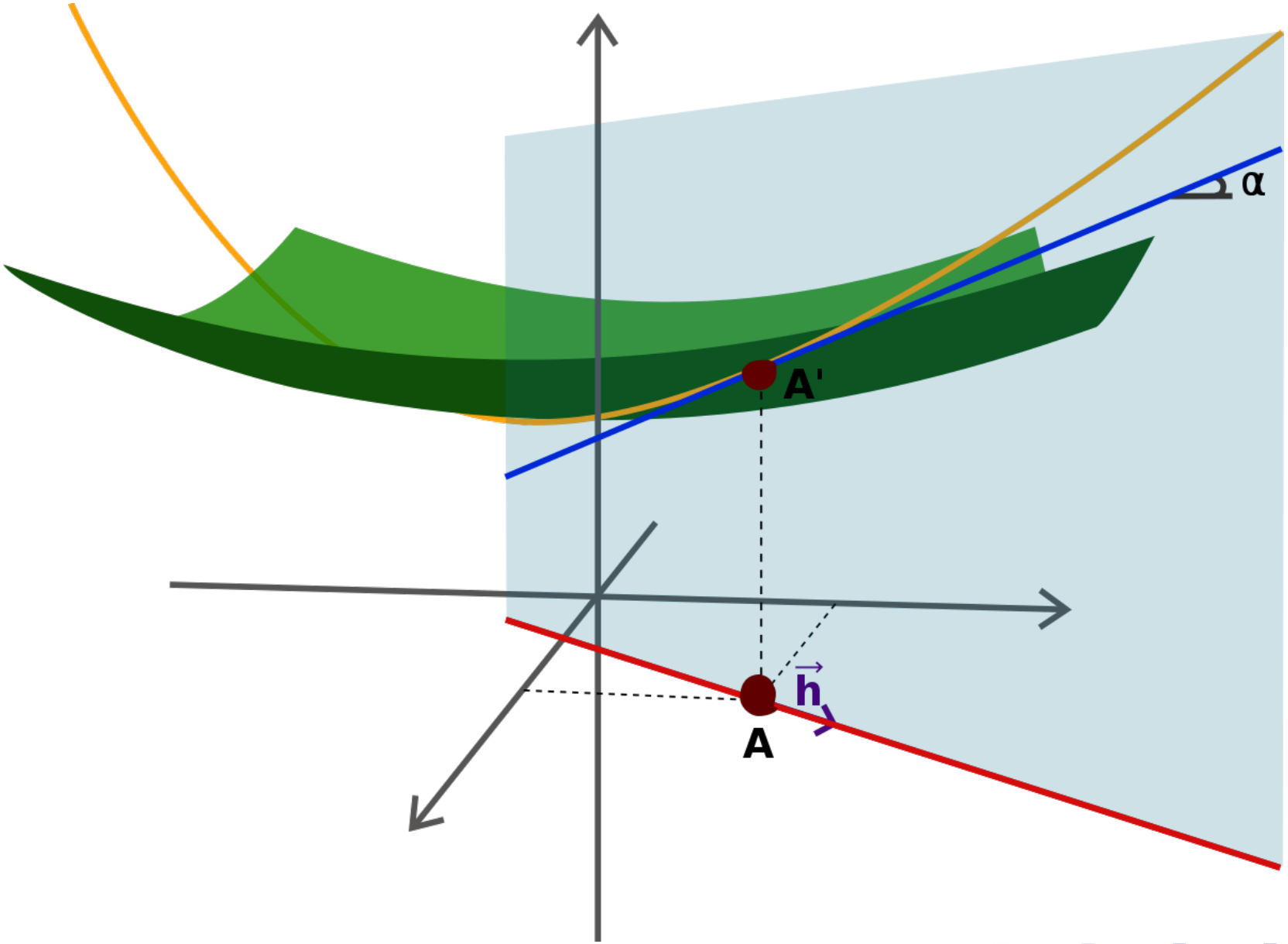
$$\lim_{t \; ! \; 0} \frac{f\,(x_0 + tv) \quad f\,(x_0)}{t} = \frac{\partial f}{\partial x_1}\,(x_0)\,v_1 + \ldots + \frac{\partial f}{\partial x_n}\,(x_0)\,v_n = r\,f\,(x_0) \quad v.$$

If the limit of right hand side exists, it is called directional derivative of f and is denoted by $\dfrac{\partial f}{\partial v}$.

Directional Derivative



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x_0 is called a saddle point of f if $r f(x_0) = 0$ but x_0 is not an local extremum point of f.

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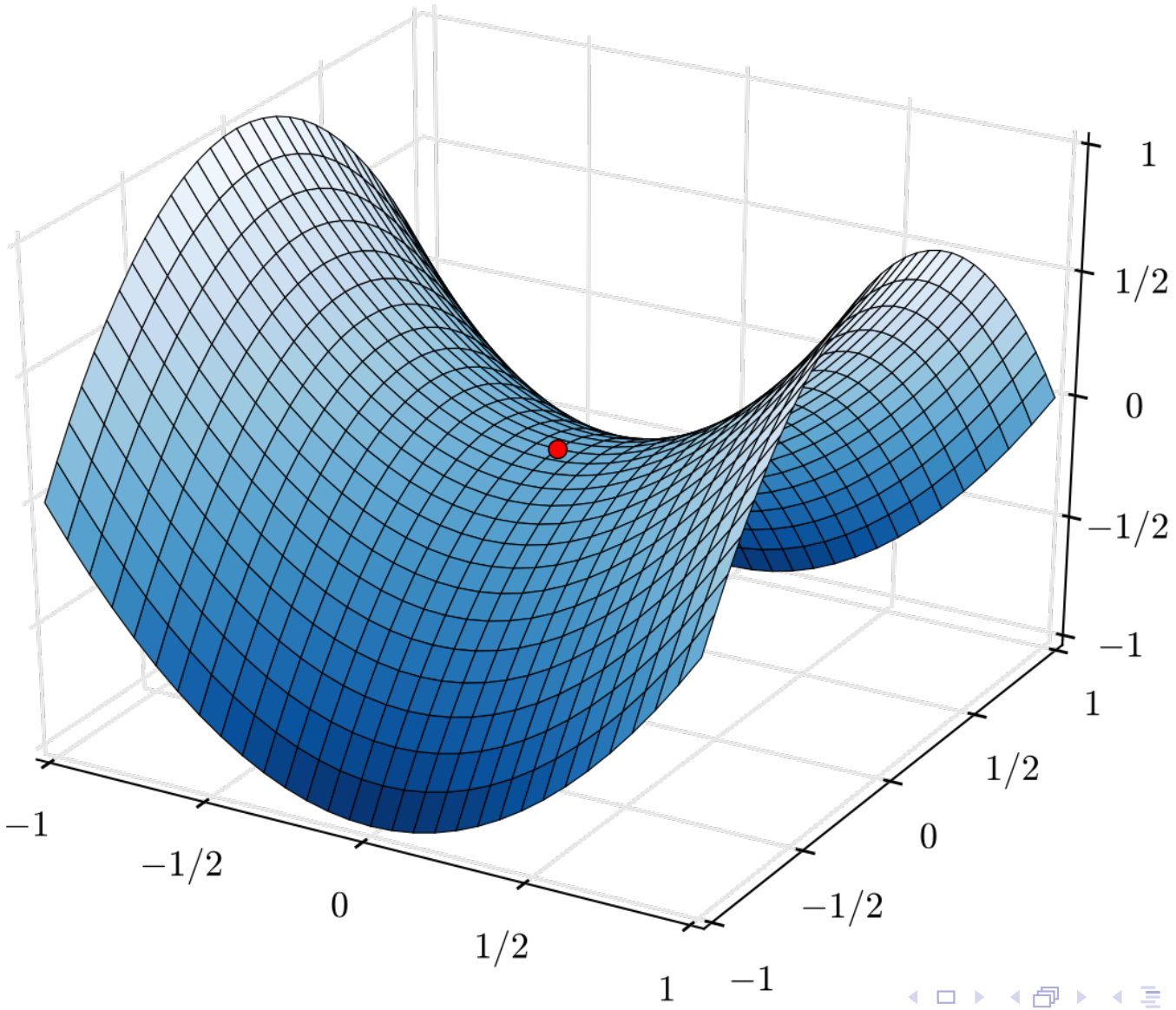

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If f is convex, then its Hesian matrix is positive semi-de nite.

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3. *if $H(x_0)$ has both positive and negative eigenvalues then x_0 is a saddle point for f.*

Gradient Descent

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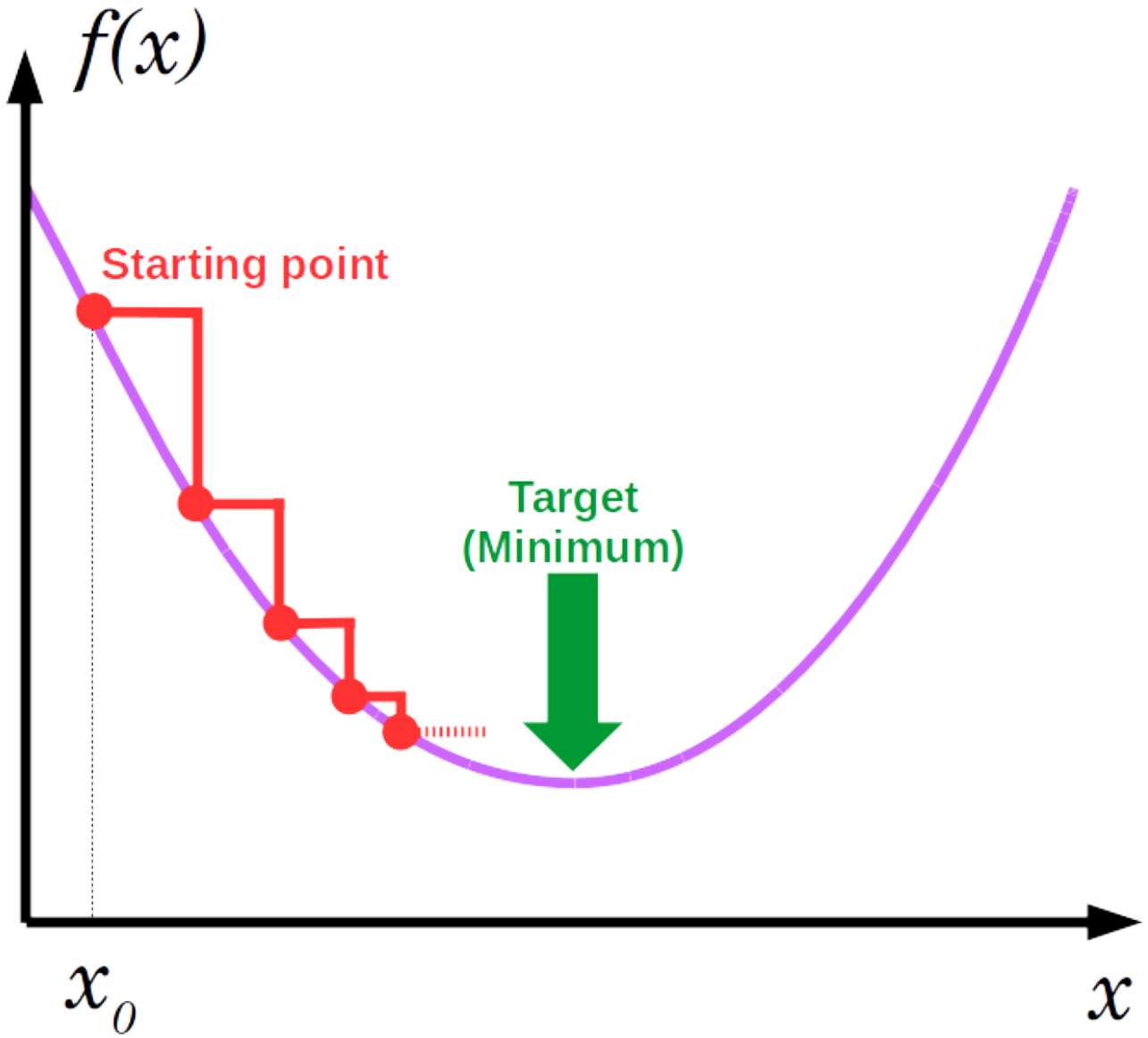
Let $f : \mathrm{R}^k \; ! \; \mathrm{R}$ be a convex function and we want to find its global minimum. This optimization algorithm is based on the fact that the fastest decreasing direction of the function is the opposite direction of gradient:

$$x_{n+1} = x_n \quad \alpha \, r \, f\,(x_n)$$

and $x_0 \; 2 \; \mathrm{R}^k$ is a arbitrary point and $\alpha > 0$.

Gradient Descent



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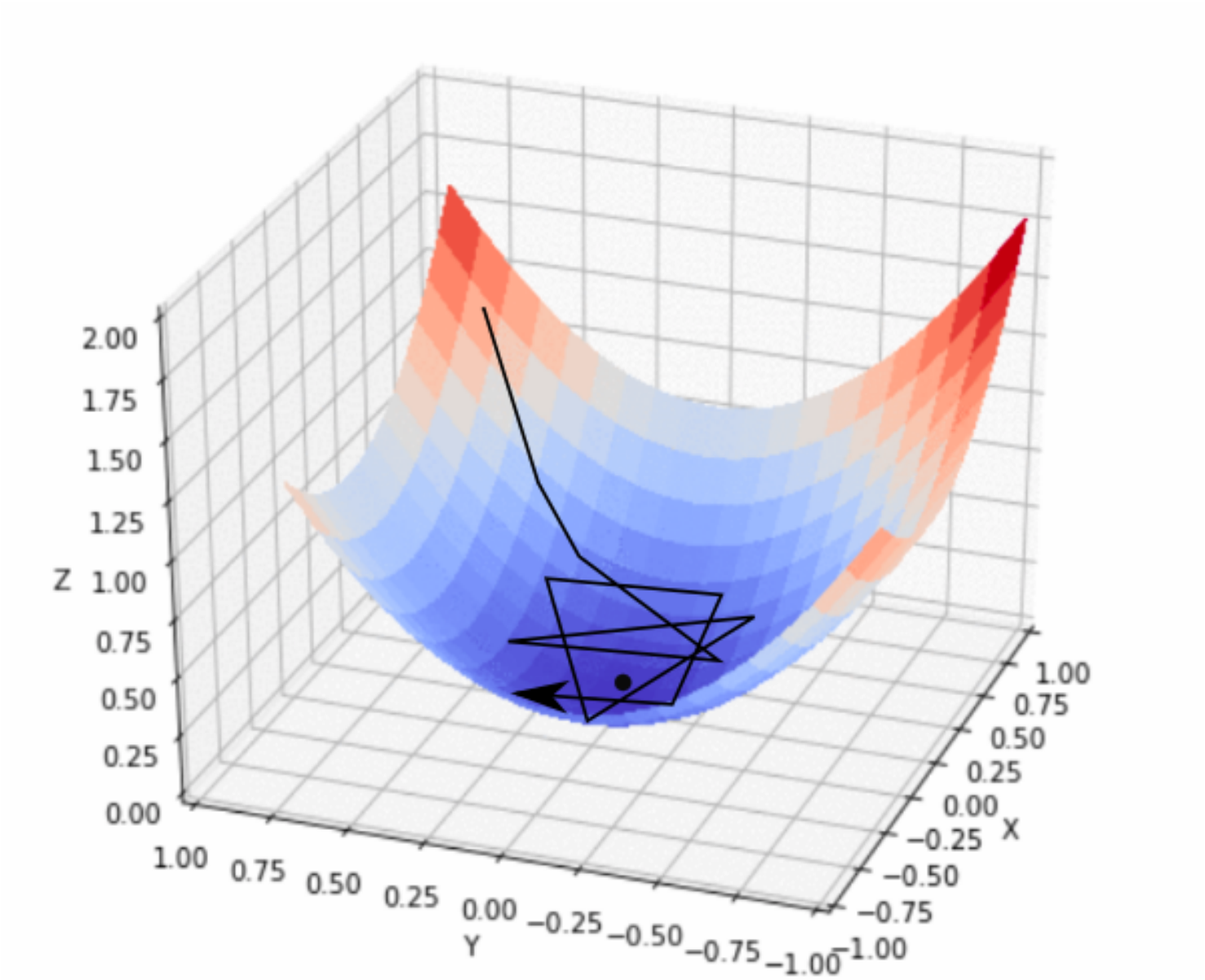

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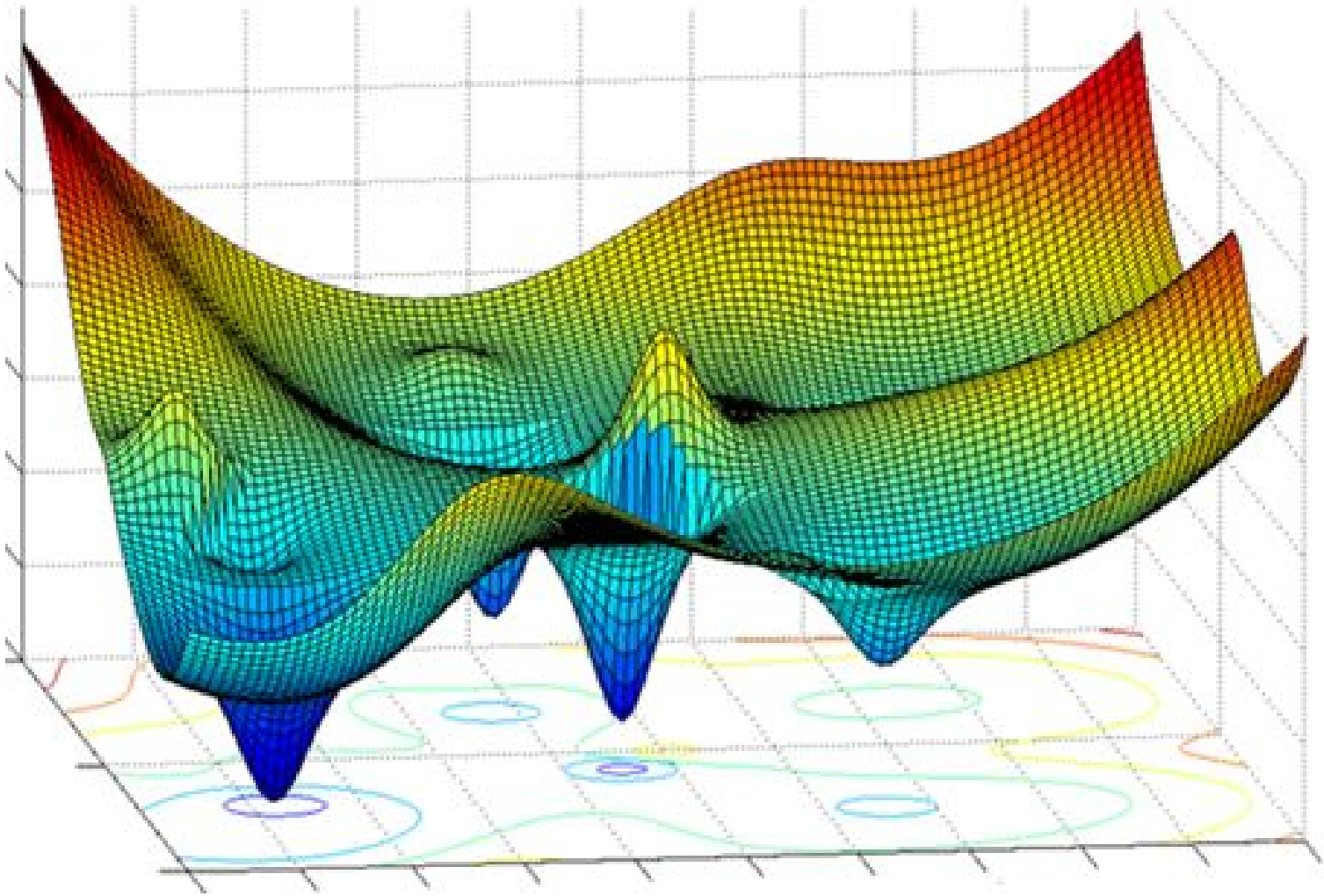

Probability

Experiment, Outcomes and the Sample Space

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- The Sample Space in this Example is: $[0, 150]$

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- *if A_n 2 F, n 2 N, then $\bigcup_{n=1}^{1} A_n$ 2 F.*

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P3. For any sequence of pairwise mutually exclusive (disjoint) events A_n 2 F, i.e., for any sequence A_n 2 F with A_i \ A_j = ? for i 6= j, we have

$$\mathrm{P}\left(\bigcup_{n=1}^{1} A_n\right) = \sum_{n=1}^{1} \mathrm{P}(A_n).$$

Probability (Measure) Definition

Probability Measure is very similar (and shares the properties of) any other Measure -

- Cardinality (no. of elements),
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The difference is only that the Probability of the Sample Space is 1, P( ) = 1.

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3. for any event A 2 F,

$$\mathrm{P}(\overline{A}) = 1 \quad \mathrm{P}(A);$$

Here $\overline{A} = A^c =$ n A.

Properties of the Probability Measure

4. If $A_1, A_2, ..., A_n$ 2 F are pairwise disjoint (mutually exclusive), i.e., if $A_i \setminus A_j = ?$ for i 6= j, then

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Properties of the Probability Measure

4. If $A_1, A_2, ..., A_n \in \mathcal{F}$ are pairwise disjoint (mutually exclusive), i.e., if $A_i \cap A_j = \emptyset$ for $i \neq j$, then

$$\mathsf{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathsf{P}(A_i);$$

5. for any events $A, B \in \mathcal{F}$ (not necessarily disjoint),

$$\mathsf{P}(A \cup B) = \mathsf{P}(A) + \mathsf{P}(B) \quad \mathsf{P}(A \cap B);$$