Mathematics for Machine Learning

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September 19, 2020

FAST DISCOVERING THE FUTURE

Signal Processing

Definition of Signal Processing

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  - A signal is mathematically just a function.
- What is a signal processing?
  - Convert one signal to another.
  - Information extraction and interpretation.

Fourier Transform

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Definition

Let $f:\mathbb{R}\to\mathbb{C}$. The Fourier transform of f is defined as follows

$$\mathcal{F}\left[f\right](x)=\hat{f}(x)=\int\limits_{-\infty}^{+\infty}f(t)\,e^{-2\pi ixt}dt.$$

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Definition

For all $p \in [1, \infty)$ denote by L^p the set of all functions f, such that

$$\|f\|_p = \left(\int_{-\infty}^{+\infty} |f(x)|^p \, dx \right)^{\frac{1}{p}} < +\infty.$$

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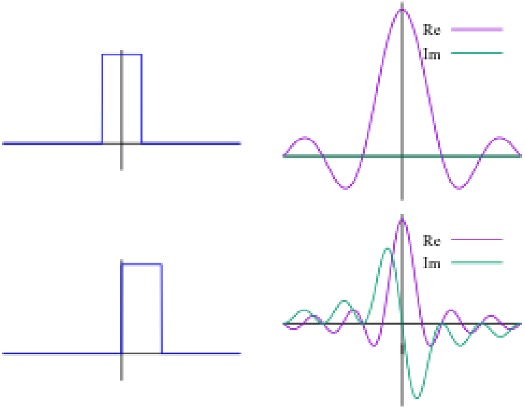
Theorem

If $f \in L^1$*, then the Fourier transform of* f *exists.*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Fourier Transform

Properties

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4. If $g=\overline{f}$, then $\hat{g}\left(x\right)=\overline{\hat{f}\left(-x\right)}$.

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Plancherel's Theorem

If $f\in L^1\cap L^2$, then $\hat{f}\in L^2$ and

$$\int\limits_{-\infty}^{+\infty}\left|\hat{f}\left(x\right)\right|^2dx=\int\limits_{-\infty}^{+\infty}\left|f\left(t\right)\right|^2dt.$$

Convolution

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Definition

Convolution of the functions $f, g : \mathbb{R} \to \mathbb{R}$, $f, g \in L^1$, *is defined as the integral of the product of the two functions after one is reversed and shifted:*

$$(f * g)(t) =: \int_{-\infty}^{+\infty} f(x)\, g(t-x)\, dx.$$

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Convolution

Young's Convolution Inequality

If $f \in L^p$, $g \in L^q$ and $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, with $1 \leq p$ and $1 \leq q \leq r < \infty$, then $\|f * g\|_r \leq \|f\|_p \|g\|_q$.

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Convolution Theorem

If $f, g \in L^1$, then $\mathcal{F}[f * g] = \mathcal{F}[f]\,\mathcal{F}[g]$.

Inverse Fourier Transform

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Let $g : \mathbb{R} \to \mathbb{C}$. *The inverse Fourier transform of* f *is defined as follows*

$$\mathcal{F}^{-1}[g](x) = \int_{-\infty}^{+\infty} g(t)\, e^{2\pi i x t} dt.$$

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Theorem

If $f, \hat{f} \in L^1$ *then* $\mathcal{F}^{-1}\left[\hat{f}\right] = f$, *almost everywhere on* $\mathbb{R}$.

Digital Signal Processing

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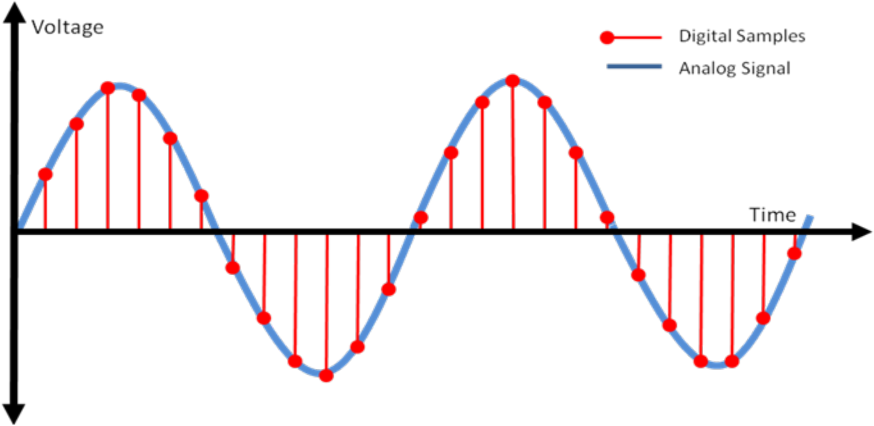
- Can be samples of a continuous signal.
- Can be discrete by nature.

Example of Sampling



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- **Audio recording:** an analog pressure wave is sampled and converted to a one-dimensional discrete-time signal.
- **Photos:** the analog scene of light is sampled using a CCD array and stored as a two-dimensional discrete-space signal
- **Ratings:** for books (Goodreads), movies (Netflix), vacation rentals (Air bnb) are stored using the integers 0-5

SP and ML Examples

- Speaker Identification

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- Speech to Text

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- Source Separation

Basic Notation

- Continuous signals:

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$$x(t) : \mathbb{R} \to \mathbb{R} \text{ or } x(t) : \mathbb{R} \to \mathbb{C}.$$

- Discrete signals:

$$x[n] : \mathbb{Z} \to \mathbb{R} \text{ or } x[n] : \mathbb{Z} \to \mathbb{C}.$$

Examples

- Delta Signal:

$$\delta_n = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0 \end{cases}$$

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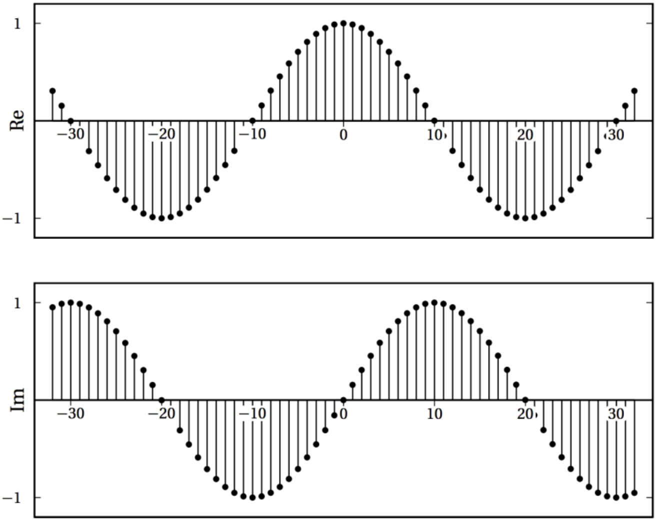
- Complex Exponential Signal:

$$x\left[n\right] = e^{i(\omega_0 n + \varphi)}$$

Complex Exponential Signal



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- ω_0 is the frequency in radians per sample.
- Frequency in cycles per sample $f = \dfrac{\omega_0}{2\pi}$.

Periodicity

Definition

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Theorem

A discrete-domain or discrete-time sinusoid is periodic if and only if its frequency ω_0 *is* π *times a rational number, that is*

$$\omega_0 = \frac{M}{N}\pi, M, N \in \mathbb{Z}.$$

Energy and Power

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$$P_x = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 .$$

Sampling

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Let us suppose that any analog signal can be represented as a sum of sinusoids of different amplitudes, frequencies and phases, i.e.

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All signals such that speech and video, lend themselves to such a represantation over any short time segment. **For speech the maximum frequency is approximately 3400 Hz.**

Sampling Theorem

Nyquist–Shannon

If the highest frequency contained in an analog signal $x_a(t)$ is F_{max} and the signal is sampled at a rate $F_s > 2F_{max}$, then $x_a(t)$ can be exactly recovered from its sample values using the interpolation function

$$g(t) = \frac{\sin(2\pi F_{max} t)}{2\pi F_{max} t}.$$

Thus $x_a(t)$ can be expressed as

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The sampling rate $F_N = 2F_{max}$ is called the Nyquist rate.

Fourier Series

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Definition

The function $T : \mathbb{R} \to \mathbb{R}$ *is called a trigonometric polynomial of degree* n*, if there exist sequences of real numbers* $\{a_k\}_{k=0}^{n}$*,* $\{b_k\}_{k=1}^{n}$ *such that* $a_n^2 + b_n^2 > 0$ *and*

$$T(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

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