Mathematics for Machine Learning

Vazgen Mikayelyan

August 8, 2020

FAST DISCOVERING THE FUTURE

Mathematical Analysis

Limit of a Sequence

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Definition

We call $x \in \mathbb{R}$ *the limit of the sequence* $\{x_n\}_{n=1}^{\infty}$ *if the following condition holds: for each real number* $\varepsilon > 0$*, there exists a natural number* n_0 *such that, for every natural number* $n \geq n_0$*, we have* $|x_n - x| < \varepsilon$*.*

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   1. $x_n + y_n \to x + y$,
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   3. $\dfrac{x_n}{y_n} \to \dfrac{x}{y}$, if $y_n \neq 0$ and $y \neq 0$.

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   4. If $x_n \leq y_n$ then $x \leq y$.

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   3. $\frac{x_n}{y_n} \to \frac{x}{y}$, if $y_n \neq 0$ and $y \neq 0$.
   4. If $x_n \leq y_n$ then $x \leq y$.
5. If $x_n, y_n \to x$ and $x_n \leq z_n \leq y_n$, then $z_n \to x$.

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If a sequence of real numbers is increasing and bounded above then it converges and if it is not bounded above, than its limit is $+\infty$.

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Theorem

If a sequence of real numbers is decreasing and bounded below then it converges and if it is not bounded below, than its limit is $-\infty$.

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Proposition

The sequence $x_n = \left(1 + \frac{1}{n}\right)^n$ *is increasing and bounded.*

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Let x_n *and* y_n *are increasing and decreasing sequences respectively. Also let* $x_n < y_n$ *for all* $n \in \mathbb{N}$ *and* $(x_n - y_n) \to 0$*. Then the sequences* x_n *and* y_n *are convergent and they have the same limit.*

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If $x_n \to x$, *then* $x_{n_k} \to x$ *where* n_k *is an arbitrary strictly increasing sequence of natural numbers.*

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c *is called a subsequential limit of the sequence* x_n *if there exists a subsequence* x_{n_k} *such that* $x_{n_k} \to c$.

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Every bounded sequence has a convergent subsequence.

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The sequence x_n is called Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ for all $n, m \geq n_0$.

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Theorem

The sequence x_n *is a Cauchy sequence if and only if it is convergent.*