Mathematics for Machine Learning: Homework 10 Deadline is 24.09.2020

September 16, 2020

1. Calculate the integral.

   a) $\iint\limits_D x \sin(x+y)\, dxdy,\ D = [0, \pi] \times \left[0, \frac{\pi}{2}\right],$

   b) $\iint\limits_D xy^2 dxdy,\ D = \left\{(x, y) : 0 \le x \le 1, x^3 \le y \le x^2\right\},$

   c) $\iint\limits_D \frac{x^2}{y^2+1} dxdy,$ where D is the domain bounded by the curves

   $$y = x, y = 0, xy = 1, x = 2.$$

2. Let X be a uniformly distributed random variable on the interval $(0, 1)$. Find the correlation of the following random variables

   a) X and X^2,

   b) X and X^3.

3. Let $X_n, n \in \mathbb{N}$ be a sequence of random variables such that their distribution function has the following forms

   $$F_{X_n}(\omega) = \begin{cases} 1 - \left(1 - \frac{1}{n}\right)^{nx}, & \text{if } x > 0 \\ 0, & \text{if } x \le 0. \end{cases}$$

   Show that X_n converges in distribution to the exponential random variable with $\lambda = 1$.

4. Let $\lambda > 0$ and $X_n, n \in \mathbb{N}$ be a sequence of random variables such that

   $$X_n \sim \textbf{Binomial}\left(n, \frac{\lambda}{n}\right)$$

   for $n > \lambda$. Show that X_n converges in distribution to $\textbf{Poisson}(\lambda)$.

5. Let $X_n, n \in \mathbb{N}$ be a sequence of random variables such that

$$X_n(\omega) = \begin{cases} n^2, \text{ with probability } \dfrac{1}{n}, \\ 0, \text{ with probability } 1 - \dfrac{1}{n}. \end{cases}$$

   Find the limit of X_n in means of the probability convergence.

6. Let X_n be a uniform random variable on the interval $\left(0, \dfrac{1}{n}\right)$, for all $n \in \mathbb{N}$. Find the limit of X_n in means of the $r - th$ mean, where $r \geq 1$.

7. Let joint PDF of random variables X and Y is

$$f(x, y) = \frac{a}{1 + x^2 + y^2 + x^2 y^2}.$$

   (a) Calculate the coefficient a.
   (b) Calculate joint CDF of random variables X and Y.
   (c) Calculate marginal CDFs of random variables X and Y.
   (d) Are X and Y independent?

8. Let joint PDF of random variables X and Y is

$$f(x, y) = \begin{cases} a\left(x^2 + \dfrac{xy}{2}\right), \text{ if } (x, y) \in [0, 1] \times [0, 2] \\ 0, \text{ otherwise.} \end{cases}$$

   (a) Calculate the coefficient a.
   (b) Calculate $\mathbb{P}(X > Y)$.
   (c) Calculate $\mathbb{P}(X > 0.5 | Y < 0.5)$.