Mathematics for Machine Learning

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FAST DISCOVERING THE FUTURE

Derivative

Definition

Let $f : X \to \mathbb{R}$, $X \subset \mathbb{R}$. *It is said* f *is differentiable at interior point* $x_0 \in X$, *if the following limit exists*

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$$f(x_0 + h) - f(x_0) = f'(x_0)\, h + o(h),\, h \to 0.$$

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Theorem

If f *has a finite derivative at* x_0 *then it is continuous at* x_0.

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4. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, if $g \neq 0$,
5. $(fg)^{(n)} = \sum_{k=0}^{n} C_n^k f^{(k)} g^{(n-k)}$.

Derivative

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Let $f : X \to \mathbb{R}$, $X \subset \mathbb{R}$. If f achieves its minimum value at interior point $x_0 \in X$ and it is differentiable x_0, then $f'(x_0) = 0$.

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Let $f : [a, b] \to \mathbb{R}$ *is continuous on* $[a, b]$, *differentiable on* (a, b) *and* $f(a) = f(b)$. *Then there exists* $c \in (a, b)$ *such that* $f'(c) = 0$.

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Derivative

Theorem

Let $f, g : [a, b] \to \mathbb{R}$ *are continuous on* $[a, b]$*, differentiable on* (a, b) *and* $g'(x_0) \neq 0, x \in (a, b)$*. Then there exists* $c \in (a, b)$ *such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

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Theorem

Let $f : X \to \mathbb{R}$ *and* X *is interval.* f *is increasing (decreasing) on* X *if and only if* $f'(x) \geq 0$ $(f'(x) \leq 0)$ *for all* $x \in X$*.*

L'Hospital's rule

Theorem

If

- *the functions* f *and* g *are differentiable in* (a, b) *and* $g(x) \neq 0, x \in (a, b)$*,*
- $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$*,*
- $\lim_{x \to a} \frac{f'(x)}{g'(x)} = K$*,*

then $\lim_{x \to a} \frac{f(x)}{g(x)} = K$*.*

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Theorem

If

- *the functions* f *and* g *are differentiable in* (a, b) *and* $g(x) \neq 0, x \in (a, b)$,
- $\lim_{x \to a} g(x) = +\infty$,
- $\lim_{x \to a} \frac{f'(x)}{g'(x)} = K$,

then $\lim_{x \to a} \frac{f(x)}{g(x)} = K$.

Extrema of Function

Definition

Let $f : X \to \mathbb{R}$ *and* x_0 *is an interior point of* X. *Then* x_0 *is called local maximum (minimum) point of* f, *if there exists* $\delta > 0$ *such that from* $x \in (x_0 - \delta, x_0 + \delta)$ *follows that* $f(x) \leq f(x_0)$ $(f(x) \geq f(x_0))$.

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Theorem

If x_0 *is an extremum point of* f *and there exists* $f'(x_0)$, *then* $f'(x_0) = 0$.

Theorem

Let f is differentiable in the intervals $(x_0 - \delta, x_0)$, $(x_0, x_0 + \delta)$ and continuous at x_0, then

1. *if $f'(x) > 0$, $x \in (x_0 - \delta, x_0)$ and $f'(x) < 0$, $x \in (x_0, x_0 + \delta)$, then x_0 is a local maximum point,*

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3. *if $f'(x)$ doesn't change it's sign then x_0 is not an extremum point.*

Extrema of Function

Theorem

Let $f'(x_0) = 0$ *and there exists finite* $f''(x_0)$, *then*

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Definition

The point x_0 is called a saddle point of function f, if there exists $\delta > 0$ such that the tangent line of the graph of the function f at the point $(x_0, f(x_0))$ lies in different sides of the graph in the intervals $(x_0 - \delta, x_0)$ and $(x_0, x_0 + \delta)$.

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Theorem

Let f *be a twice differentiable function at* x_0*. If there exists* $\delta > 0$ *such that* f'' *has different signs in the intervals* $(x_0 - \delta, x_0)$ *and* $(x_0, x_0 + \delta)$*, then* x_0 *is a saddle point of function* f*.*

Convex Functions

Definition

Let $f : X \to \mathbb{R}$ *and* $X \subset \mathbb{R}$ *is an interval. The function* f *is called convex if*

$$f\left(\alpha x + (1-\alpha)\,y\right) \leq \alpha f\left(x\right) + (1-\alpha)\,f\left(y\right)$$

for all $x, y \in X$ *and* $\alpha \in [0,1]$.

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Theorem

Let $f : X \to \mathbb{R}$ *and* $X \subset \mathbb{R}$ *is an interval* f *is twice differentiable.* f *is convex if and only if* $f'' \geq 0$.

Convex Functions

Theorem

Let $f : X \to \mathbb{R}$ *and* $X \subset \mathbb{R}$ *is an interval. If* f *is a convex function, then*

$$f\left(\alpha_1 x_1 + \ldots + \alpha_n x_n\right) \leq \alpha_1 f\left(x_1\right) + \ldots + \alpha_n f\left(x_n\right),$$

for all $x_i \in X$, $\alpha_i \in [0, 1]$, $1 \leq i \leq n$ *such that* $\sum_{i=1}^{n} \alpha_i = 1.$

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The series $\sum_{n=1}^{\infty} a_n$ is called convergent if A is finite, otherwise it is called divergent.