Mathematics for Machine Learning

Lab 9

Problem 1. Suppose X is a random variable taking values $\{1, 2, 3, 4\}$ with equal probabilities. Derive the CDF of this random variable.

Problem 2. Let $\mathbb{E}(2^X) = 4$. Show that $\mathbb{P}(X \geq 3) \leq \frac{1}{2}$.

Problem 3. Suppose that the amount of time one spends in a bank is exponentially distributed with mean 10 minutes. What is the probability that a customer will spend more than fifteen minutes in the bank? What is the probability that a customer will spend more than fifteen minutes in the bank given that she is still in the bank after ten minutes?

Problem 4. An insurance company receives on average 8 claims per day. What is the probability of getting

1) no claims in the next day;

2) at least 5 claims in the next day?

Problem 5. Suppose X takes only non-negative integer values. Show that

$$\mathbb{E}X = \sum_{k=1}^{\infty} \mathbb{P}(X \geq k).$$

Problem 6. Let $X \geq 0$ and $\mathbb{E}X < \infty$. Show that

$$\mathbb{E}X = \int_0^{\infty} (1 - F(x))dx.$$

Problem 7. Find the limit of the following functional sequences and provide the domain of convergence.

1) $f_n(x) = x^n$

2) $f_n(x) = \left(\dfrac{2x}{1 + x^2}\right)^n$

3) $f_n(x) = \dfrac{nx^2}{n + 1}$

Problem 8. Provide the domain of convergence of the following functional series

1) $\displaystyle\sum_{n=1}^{\infty} \ln^n x$ **2)** $\displaystyle\sum_{n=1}^{\infty} ne^{-nx}$

Problem 9. Discuss the uniform convergence of the following functional sequences on the given sets

1) $f_n(x) = x^n$, when $x \in [0; 0.99]$ and $x \in [0, 1)$

2) $f_n(x) = \dfrac{x}{x+n}$, when $x \in [0; 100]$ and $x \in [0, +\infty)$

3) $f_n(x) = \dfrac{x^4 + nx}{n}$, when $x \in \mathbb{R}$

Problem 10. Discuss the uniform convergence of the following functional series on the given set

1) $\displaystyle\sum_{n=1}^{\infty} \dfrac{nx}{1+n^5 x^2}, \quad x \in \mathbb{R}$ **2)** $\displaystyle\sum_{n=1}^{\infty} n^{10} e^{-nx^2}, \quad |x| \geq \delta > 0$