Mathematics for Machine Learning

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FAST DISCOVERING THE FUTURE

Differential

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f is called differentiable at the point (x_0, y_0) if there exists $A, B \in \mathbb{R}$ such that

$$f(x_0 + \Delta x, y_0 + \Delta y) = f(x_0, y_0) + A\Delta x + B\Delta y + o(\rho), \rho \to 0,$$

where $\rho = \sqrt{\Delta x^2 + \Delta y^2}$.

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Theorem

If partial derivatives of the first degree of f are continuous at (x_0, y_0) then it is differentiable at (x_0, y_0). The inverse is not true.

Differential

Example

$$f(x,y) = \begin{cases} x^2 \sin \dfrac{1}{x}, \ \textit{if } (x,y) \neq (0,0), \\ 0, \ \textit{if } (x,y) = (0,0) \end{cases}$$

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Definition

$$df(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)\,\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\,\Delta y \ \textit{is called differential of } f.$$

Directional Derivative

Theorem

Let $f : X \to \mathbb{R}$, $X \subset \mathbb{R}^n$ and x_0 is an interior point of X. If f is differentiable at x_0, then for all $v \in \mathbb{R}^n$ such that $\|v\| = 1$ we have

$$\lim_{t \to 0} \frac{f(x_0 + tv) - f(x_0)}{t} = \frac{\partial f}{\partial x_1}(x_0)\, v_1 + \ldots + \frac{\partial f}{\partial x_n}(x_0)\, v_n$$

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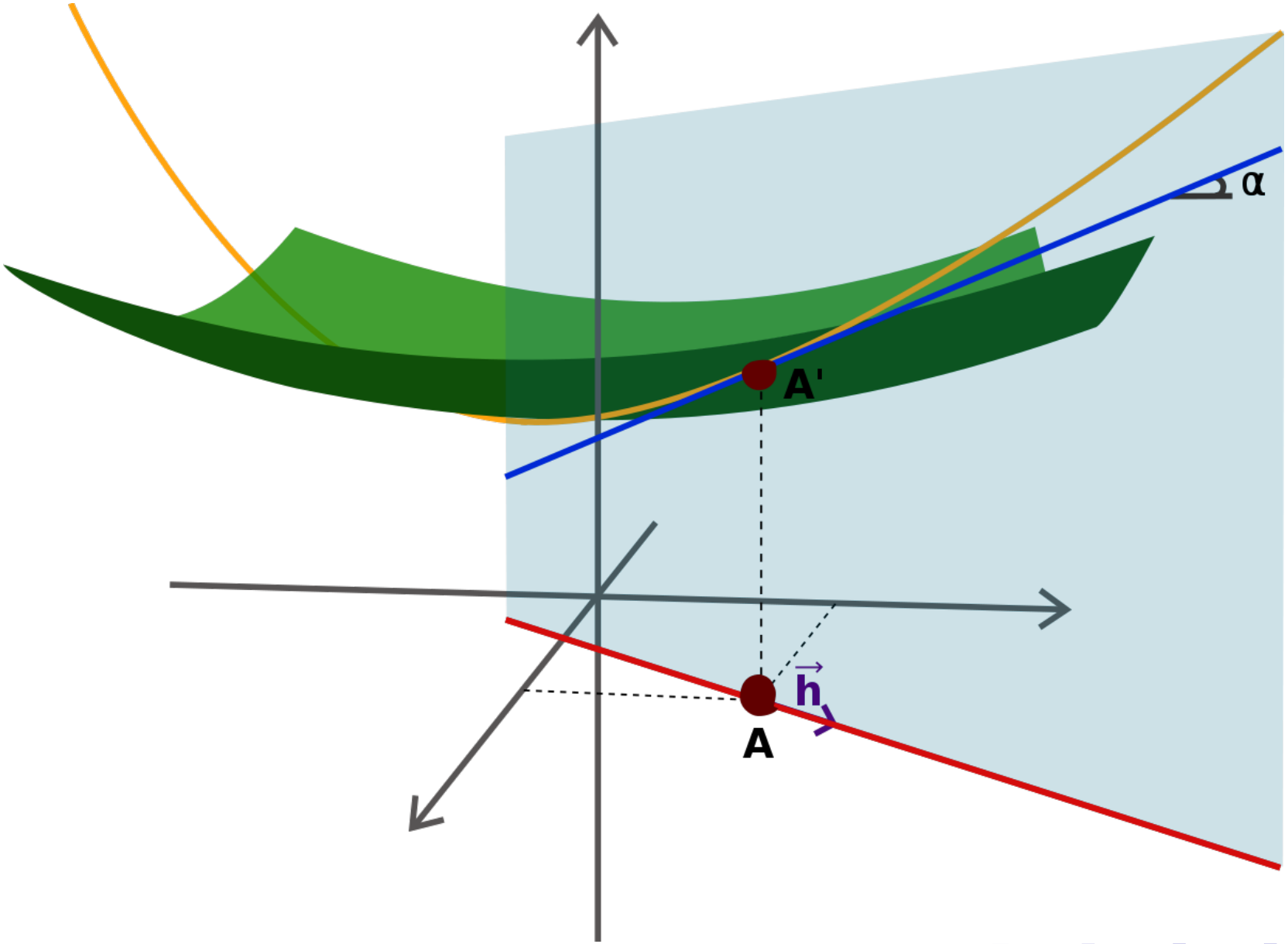
If the limit of right hand side exists, it is called directional derivative of f and is denoted by $\frac{\partial f}{\partial v}$.

Directional Derivative



Extrema of Functions of Many Variables

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x_0 is called a saddle point of f if $\nabla f(x_0) = 0$ but x_0 is not an local extremum point of f.

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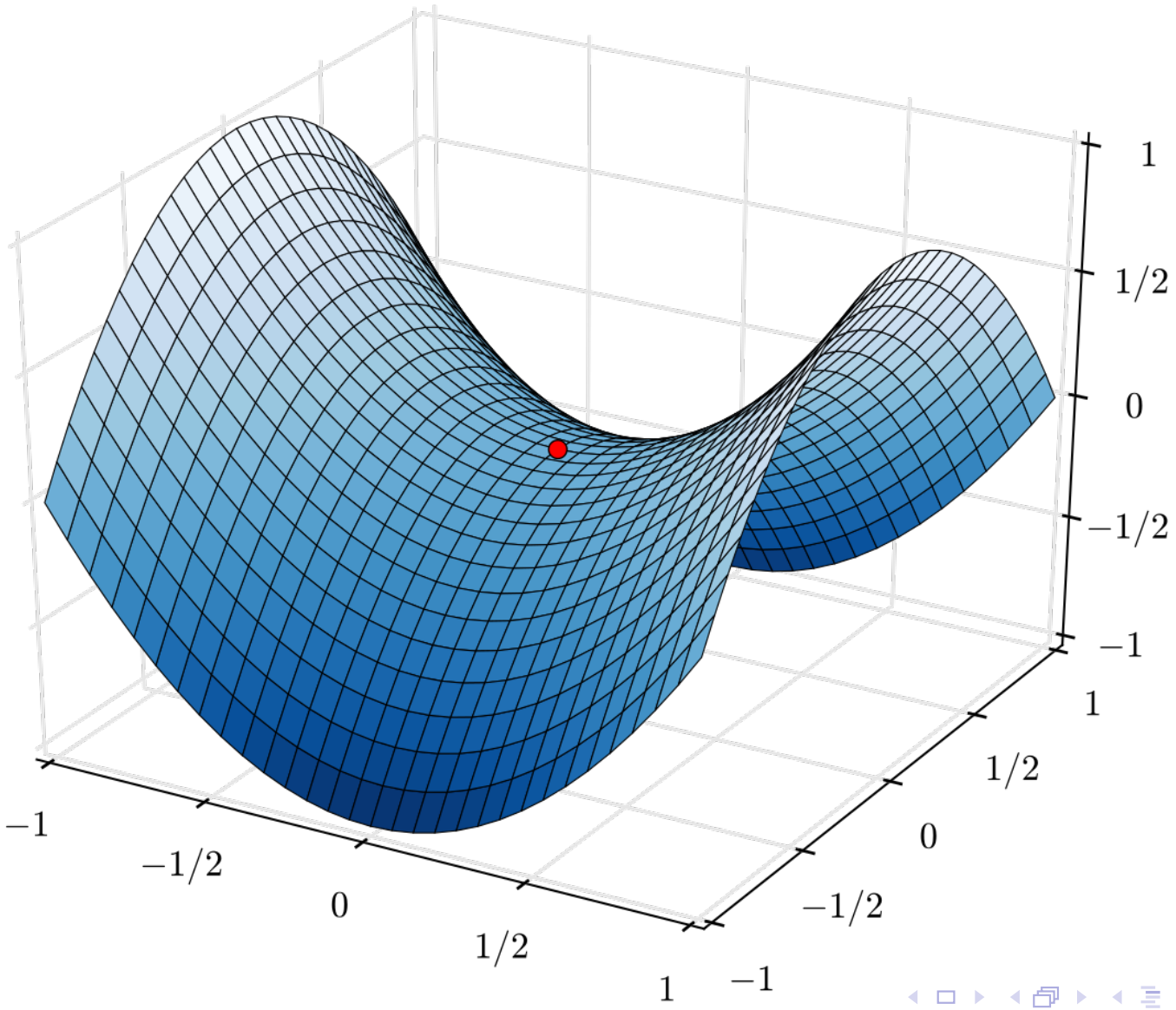

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Let $f : X \to \mathbb{R}$ and $X \subset \mathbb{R}^n$. If all second partial derivatives of f exist and are continuous over the domain of the function, then the matrix $H_{i,j} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$, $i, j = 1, \ldots, n$ is called the Hesian matrix of f.

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If f is convex, then its Hesian matrix is positive semi-definite.

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2. *if $H(x_0)$ is negative definite, then f attains a local maximum at x_0,*
3. *if $H(x_0)$ has both positive and negative eigenvalues then x_0 is a saddle point for f.*

Gradient Descent

Let $f : \mathbb{R}^k \to \mathbb{R}$ be a convex function and we want to find its global minimum.

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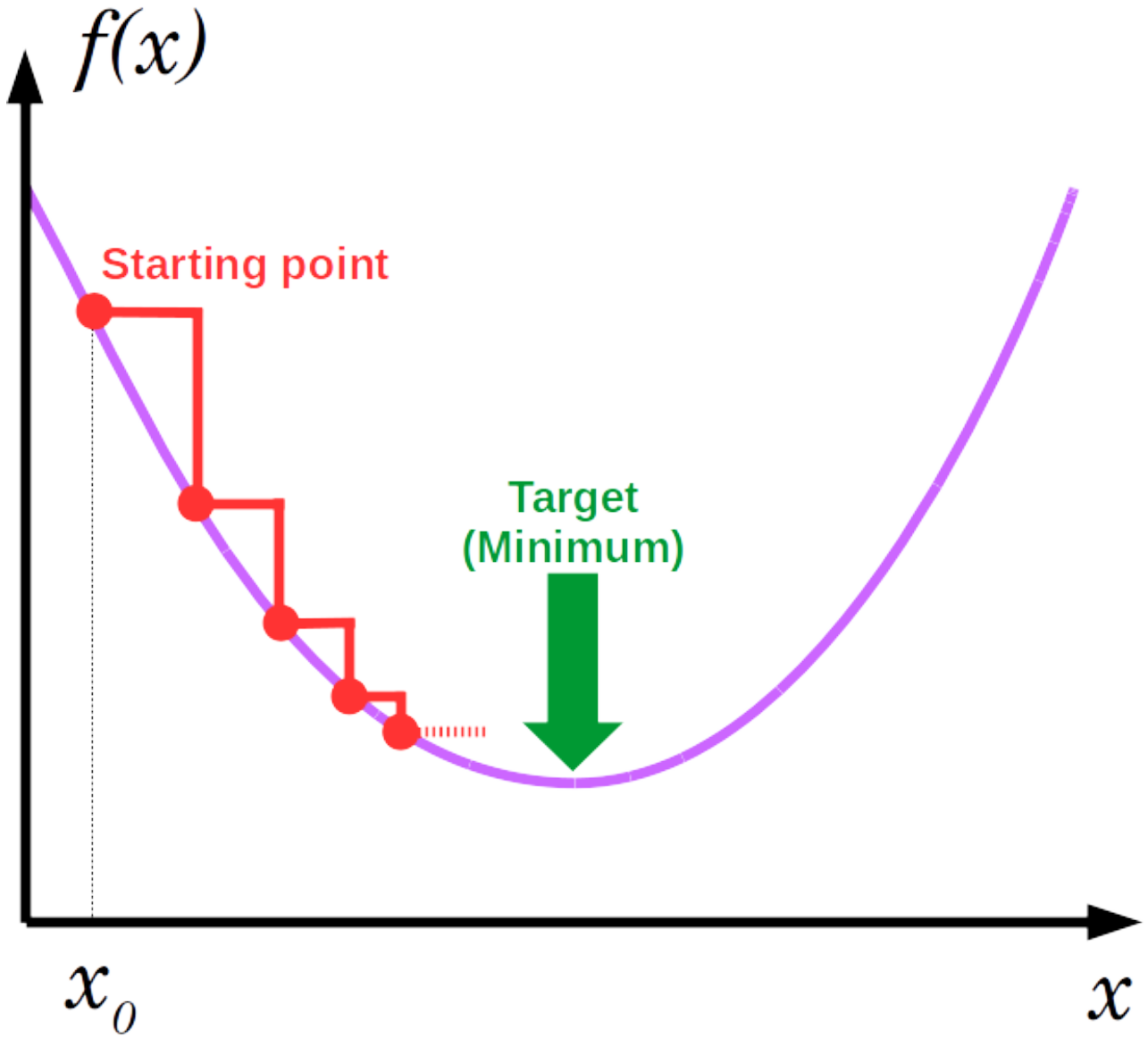
Let $f : \mathbb{R}^k \to \mathbb{R}$ be a convex function and we want to find its global minimum. This optimization algorithm is based on the fact that the fastest decreasing direction of the function is the opposite direction of gradient:

$$x_{n+1} = x_n - \alpha \nabla f\left(x_n\right)$$

and $x_0 \in \mathbb{R}^k$ is a arbitrary point and $\alpha > 0$.

Gradient Descent



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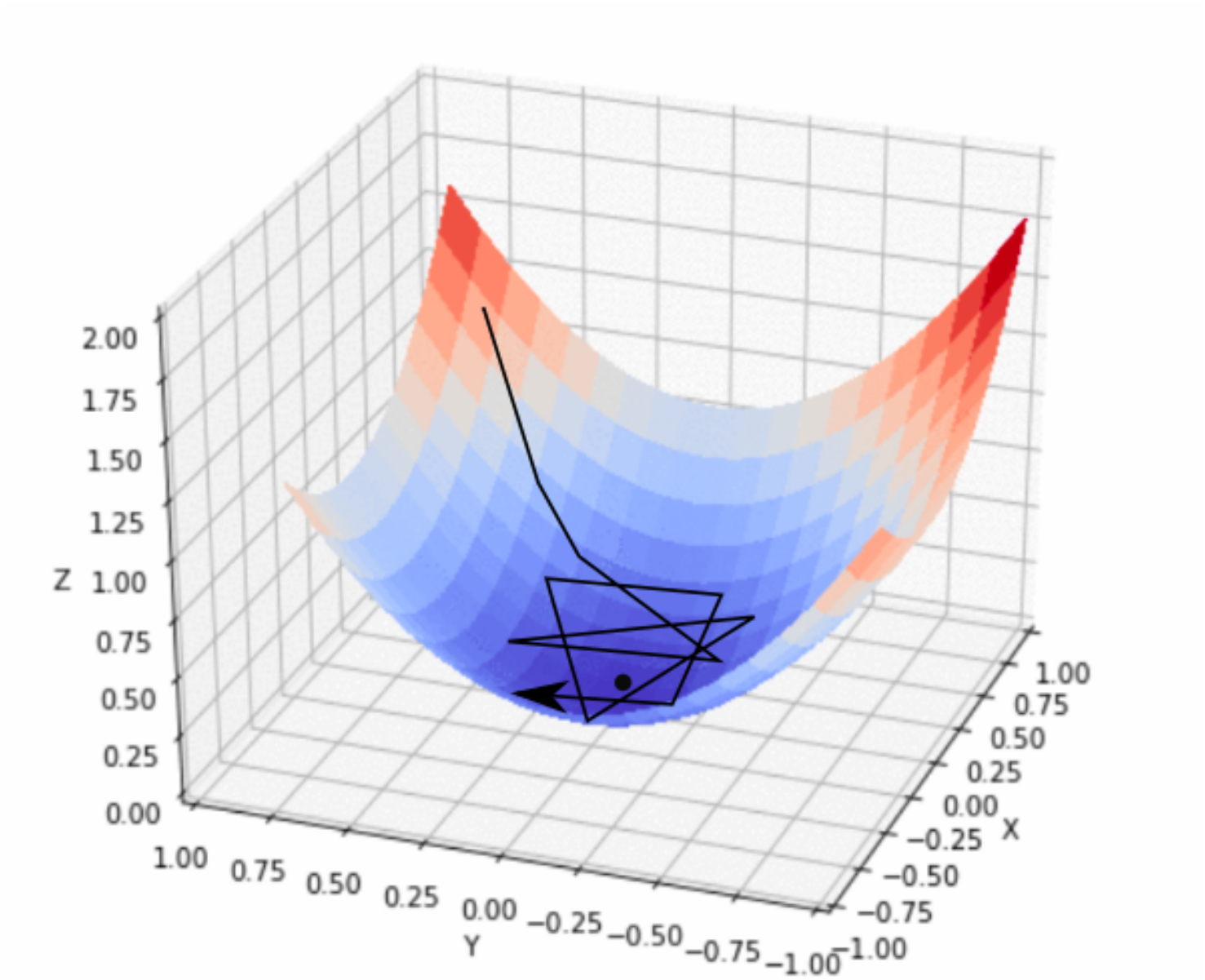

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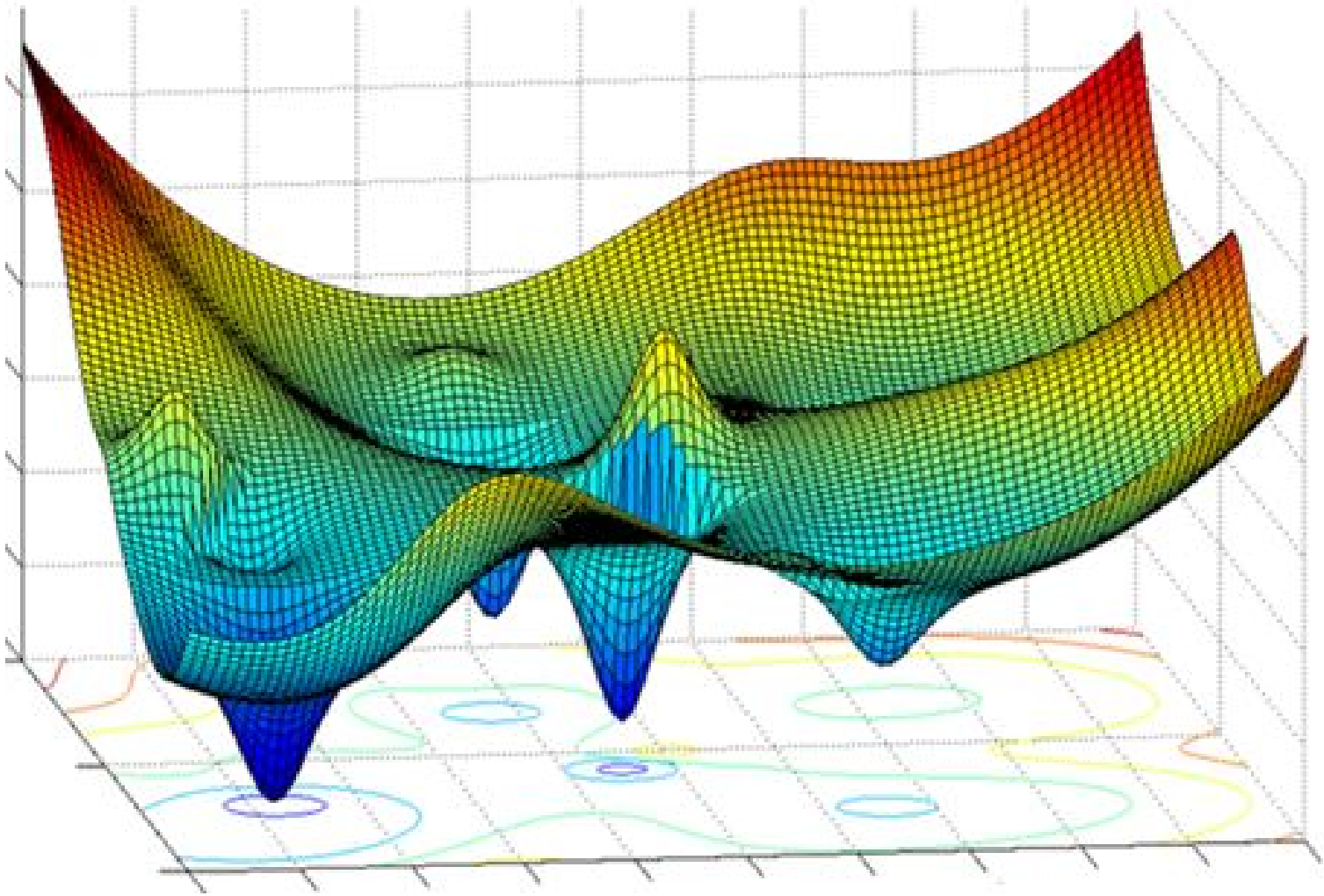

Probability

Experiment, Outcomes and the Sample Space

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- *if* $A_n \in \mathcal{F}$, $n \in \mathbb{N}$, *then* $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

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P2. $\mathbb{P}(\Omega) = 1$;

P3. For any sequence of pairwise mutually exclusive (disjoint) events $A_n \in \mathcal{F}$, i.e., for any sequence $A_n \in \mathcal{F}$ with $A_i \cap A_j = \varnothing$ for $i \neq j$, we have

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

Probability (Measure) Definition

Probability Measure is very similar (and shares the properties of) any other Measure -

- Cardinality (no. of elements),
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- Area (in 2D),
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The difference is only that the Probability of the Sample Space is 1, $\mathbb{P}(\Omega) = 1$.

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3. for any event $A \in \mathcal{F}$,

$$\mathbb{P}(\overline{A}) = 1 - \mathbb{P}(A);$$

Here $\overline{A} = A^c = \Omega \setminus A$.

Properties of the Probability Measure

4. If $A_1, A_2, ..., A_n \in \mathcal{F}$ are pairwise disjoint (mutually exclusive), i.e., if $A_i \cap A_j = \varnothing$ for $i \neq j$, then

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5. for any events $A, B \in \mathcal{F}$ (not necessarily disjoint),

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B);$$