Mathematics for Machine Learning

Vazgen Mikayelyan

September 22, 2020

FAST DISCOVERING THE FUTURE

Fourier Series

Fourier Series

Definition

The function $T : \mathbb{R} \to \mathbb{R}$ *is called a trigonometric polynomial of degree* n*, if there exist sequences of real numbers* $\{a_k\}_{k=0}^{n}$*,* $\{b_k\}_{k=1}^{n}$ *such that* $a_n^2 + b_n^2 > 0$ *and*

$$T(x) = a_0 + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx).$$

Some Identities

1. $$\int_{-\pi}^{\pi} \sin mx dx = 0,\ m \in \mathbb{Z},$$

Some Identities

1. $\int\limits_{-\pi}^{\pi} \sin mx dx = 0,\ m \in \mathbb{Z},$
2. $\int\limits_{-\pi}^{\pi} \cos mx dx = \begin{cases} 0, m \in \mathbb{Z} \setminus \{0\} \\ 2\pi, m = 0 \end{cases} \quad ,$

Some Identities

1. $$\int_{-\pi}^{\pi} \sin mx dx = 0, \ m \in \mathbb{Z},$$
2. $$\int_{-\pi}^{\pi} \cos mx dx = \begin{cases} 0, m \in \mathbb{Z} \setminus \{0\} \\ 2\pi, m = 0 \end{cases},$$
3. $$\int_{-\pi}^{\pi} \sin mx \cos kx dx = 0, \ m, k \in \mathbb{N},$$

Some Identities

1. $\int_{-\pi}^{\pi} \sin mx dx = 0, \; m \in \mathbb{Z},$
2. $\int_{-\pi}^{\pi} \cos mx dx = \begin{cases} 0, m \in \mathbb{Z} \setminus \{0\} \\ 2\pi, m = 0 \end{cases},$
3. $\int_{-\pi}^{\pi} \sin mx \cos kx dx = 0, \; m, k \in \mathbb{N},$
4. $\int_{-\pi}^{\pi} \sin mx \sin kx dx = \begin{cases} 0, m \neq k \\ \pi, m = k \end{cases}, \; m, k \in \mathbb{N},$

Some Identities

1. $$\int_{-\pi}^{\pi} \sin mx dx = 0,\ m \in \mathbb{Z},$$
2. $$\int_{-\pi}^{\pi} \cos mx dx = \begin{cases} 0, m \in \mathbb{Z} \setminus \{0\} \\ 2\pi, m = 0 \end{cases},$$
3. $$\int_{-\pi}^{\pi} \sin mx \cos kx dx = 0,\ m, k \in \mathbb{N},$$
4. $$\int_{-\pi}^{\pi} \sin mx \sin kx dx = \begin{cases} 0, m \neq k \\ \pi, m = k \end{cases},\ m, k \in \mathbb{N},$$
5. $$\int_{-\pi}^{\pi} \cos mx \cos kx dx = \begin{cases} 0, m \neq k \\ \pi, m = k \end{cases},\ m, k \in \mathbb{N}.$$

Trigonometric Series

Theorem

If the trigonometric series $T(x) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos nx + b_n \sin nx)$ *is uniformly convergent on* $[-\pi, \pi]$, *then*

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} T(x) \cos kx dx, k \in \mathbb{N} \cup \{0\},$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} T(x) \sin kx dx, k \in \mathbb{N}.$$

Fourier Series

Definition

Let f *be a* 2π *periodic function and it is integrable on* $[-\pi, \pi]$ *and*

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ktdt, b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ktdt, k \in \mathbb{N} \cup \{0\}.$$

a_k *and* b_k *are called Fourier coefficients of* f *and the trigonometric series*

$$\frac{a_0}{2} + \sum_{n=1}^{+\infty}(a_k \cos kx + b_k \sin kx)$$

is called Fourier series of f.

Fourier Series

Let $x_0 \in \mathbb{R}$ f has finite limits $f\left(x_0-\right)$ and $f\left(x_0+\right)$. Denote

$$S_0 = \frac{f\left(x_0-\right) + f\left(x_0+\right)}{2}$$

and $\varphi\left(t\right) = f\left(x_0 + t\right) + f\left(x_0 - t\right) - 2S_0$.

Let $x_0 \in \mathbb{R}$ f has finite limits $f\left(x_0-\right)$ and $f\left(x_0+\right)$. Denote

$$S_0 = \frac{f\left(x_0-\right) + f\left(x_0+\right)}{2}$$

and $\varphi\left(t\right) = f\left(x_0 + t\right) + f\left(x_0 - t\right) - 2S_0$.

Dini's Theorem

If there exists $\delta \in (0, \pi)$ such that

$$\int_0^\delta \frac{\left|\varphi\left(t\right)\right|}{t} dt < +\infty,$$

then the Fourier series of f converges to S_0 at x_0.

Lipschitz's Theorem

If f is continuous at x_0 and for all $t \in (-\delta, \delta)$ holds

$$|f(x_0 + t) - f(x_0)| \leq L\,|t|^{\alpha},$$

for some $\alpha \in (0, 1]$, then the Fourier series of f converges to $f(x_0)$ at x_0.

Fejer's Theorem

Let $f \in C(\mathbb{R})$ be 2π periodic function and S_n is the n-th partial sum of the Fourier series of f, then $\sigma_n(x) \rightrightarrows f(x), x \in \mathbb{R}$, where

$$\sigma_n(x) = \frac{S_0(x) + \ldots + S_{n-1}(x)}{n}, x \in \mathbb{R}, n \in \mathbb{N}.$$

Complex Spaces

Definition

Let V be a complex vector space.

Complex Spaces

Definition

Let V be a complex vector space. A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is called **inner product on V**, *if*

Complex Spaces

Definition

Let V *be a complex vector space. A function* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ *is called* **inner product on** V**,** *if*

- $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ *(conjugate symmetric)*

Complex Spaces

Definition

Let V *be a complex vector space. A function* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ *is called* **inner product on** V**,** *if*

- $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ *(conjugate symmetric)*
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ *and* $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ *iff* $\mathbf{x} = 0$

Complex Spaces

Definition

Let V be a complex vector space. A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is called **inner product on V**, *if*

- $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ *(conjugate symmetric)*
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ *and* $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ *iff* $\mathbf{x} = 0$
- $\langle a\mathbf{x}_1 + b\mathbf{x}_2, \mathbf{y} \rangle = a\langle \mathbf{x}_1, \mathbf{y} \rangle + b\langle \mathbf{x}_2, \mathbf{y} \rangle$

Complex Spaces

Definition

Let V *be a complex vector space. A function* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ *is called* **inner product on** V, *if*

- $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ *(conjugate symmetric)*
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ *and* $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ *iff* $\mathbf{x} = 0$
- $\langle a\mathbf{x}_1 + b\mathbf{x}_2, \mathbf{y} \rangle = a\langle \mathbf{x}_1, \mathbf{y} \rangle + b\langle \mathbf{x}_2, \mathbf{y} \rangle$

It easy to see that

$$\langle \mathbf{x}, a\mathbf{y}_1 + b\mathbf{y}_2 \rangle = \overline{a}\langle \mathbf{x}, \mathbf{y}_1 \rangle + \overline{b}\langle \mathbf{x}, \mathbf{y}_2 \rangle.$$

Complex Spaces

Definition

Let V be a complex vector space. A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is called **inner product on V**, *if*

- $\langle \mathbf{x}, \mathbf{y} \rangle = \overline{\langle \mathbf{y}, \mathbf{x} \rangle}$ *(conjugate symmetric)*
- $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ *and* $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ *iff* $\mathbf{x} = 0$
- $\langle a\mathbf{x}_1 + b\mathbf{x}_2, \mathbf{y} \rangle = a\langle \mathbf{x}_1, \mathbf{y} \rangle + b\langle \mathbf{x}_2, \mathbf{y} \rangle$

It easy to see that

$$\langle \mathbf{x}, a\mathbf{y}_1 + b\mathbf{y}_2 \rangle = \overline{a}\langle \mathbf{x}, \mathbf{y}_1 \rangle + \overline{b}\langle \mathbf{x}, \mathbf{y}_2 \rangle.$$

Proposition

Prove that $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$ *is an inner product on* $\mathbb{C}^n$.

Fourier Basis

An important orthonormal basis for C^N, is the normalized Fourier basis defined as

$$\omega_m[n] = \frac{1}{\sqrt{N}} e^{i\frac{2\pi}{N}nm}, n, m \in \{0, 1, \ldots, N-1\}.$$

Fourier Basis

An important orthonormal basis for C^N, is the normalized Fourier basis defined as

$$\omega_m\left[n\right] = \frac{1}{\sqrt{N}} e^{i\frac{2\pi}{N}nm}, n, m \in \left\{0, 1, \ldots, N-1\right\}.$$

It easy to see that

$$\langle \omega_i, \omega_j \rangle = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

Discrete Fourier Transform

Let $\{x_n\}_{n=0}^{N-1}$ be a finite sequence of complex numbers.

Discrete Fourier Transform

Let $\{x_n\}_{n=0}^{N-1}$ be a finite sequence of complex numbers. Denote $x = [x_0, x_1, \ldots, x_{N-1}]^T \in \mathbb{C}^N$.

Discrete Fourier Transform

Let $\{x_n\}_{n=0}^{N-1}$ be a finite sequence of complex numbers. Denote $x = [x_0, x_1, \ldots, x_{N-1}]^T \in \mathbb{C}^N$. Since $\{\omega_i\}_{i=0}^{N-1}$ is a basis for $\mathbb{C}^N$, thus there exist $X_k \in \mathbb{C}, k = 0, \ldots, N-1$, such that

$$x = \sum_{k=0}^{N-1} X_k \omega_k$$

Discrete Fourier Transform

Let $\{x_n\}_{n=0}^{N-1}$ be a finite sequence of complex numbers. Denote $x = [x_0, x_1, \dots, x_{N-1}]^T \in \mathbb{C}^N$. Since $\{\omega_i\}_{i=0}^{N-1}$ is a basis for $\mathbb{C}^N$, thus there exist $X_k \in \mathbb{C}, k = 0, \dots, N-1$, such that

$$x = \sum_{k=0}^{N-1} X_k \omega_k$$

or equivalently

$$x_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X_n e^{i \frac{2\pi}{N} nk}, k = 0, \dots, N-1.$$

Discrete Fourier Transform

Let $\{x_n\}_{n=0}^{N-1}$ be a finite sequence of complex numbers. Denote $x = [x_0, x_1, \ldots, x_{N-1}]^T \in \mathbb{C}^N$. Since $\{\omega_i\}_{i=0}^{N-1}$ is a basis for $\mathbb{C}^N$, thus there exist $X_k \in \mathbb{C}, k = 0, \ldots, N-1$, such that

$$x = \sum_{k=0}^{N-1} X_k \omega_k$$

or equivalently

$$x_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X_n e^{i\frac{2\pi}{N}nk}, k = 0, \ldots, N-1.$$

Note that for all $l \in \{0, \ldots, N-1\}$

$$\langle x, \omega_l \rangle =$$

Discrete Fourier Transform

Let $\{x_n\}_{n=0}^{N-1}$ be a finite sequence of complex numbers. Denote $x = [x_0, x_1, \ldots, x_{N-1}]^T \in \mathbb{C}^N$. Since $\{\omega_i\}_{i=0}^{N-1}$ is a basis for $\mathbb{C}^N$, thus there exist $X_k \in \mathbb{C}, k = 0, \ldots, N-1$, such that

$$x = \sum_{k=0}^{N-1} X_k \omega_k$$

or equivalently

$$x_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X_n e^{i\frac{2\pi}{N}nk}, k = 0, \ldots, N-1.$$

Note that for all $l \in \{0, \ldots, N-1\}$

$$\langle x, \omega_l \rangle = \sum_{k=0}^{N-1} X_k \langle \omega_k, \omega_l \rangle$$

Discrete Fourier Transform

Let $\{x_n\}_{n=0}^{N-1}$ be a finite sequence of complex numbers. Denote $x = [x_0, x_1, \dots, x_{N-1}]^T \in \mathbb{C}^N$. Since $\{\omega_i\}_{i=0}^{N-1}$ is a basis for $\mathbb{C}^N$, thus there exist $X_k \in \mathbb{C}, k = 0, \dots, N-1$, such that

$$x = \sum_{k=0}^{N-1} X_k \omega_k$$

or equivalently

$$x_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X_n e^{i\frac{2\pi}{N}nk}, k = 0, \dots, N-1.$$

Note that for all $l \in \{0, \dots, N-1\}$

$$\langle x, \omega_l \rangle = \sum_{k=0}^{N-1} X_k \langle \omega_k, \omega_l \rangle = X_l.$$

Discrete Fourier Transform

Definition

The discrete Fourier transform (DFT) of the complex sequence $\{x_n\}_{n=0}^{N-1}$, *is the following sequence*

$$X_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_n e^{-i\frac{2\pi}{N}nk}, k = 0, \ldots, N-1.$$

Discrete Fourier Transform

Definition

The discrete Fourier transform (DFT) of the complex sequence $\{x_n\}_{n=0}^{N-1}$, is the following sequence

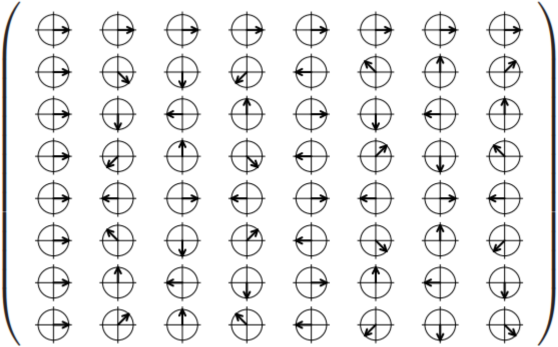
$$X_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x_n e^{-i\frac{2\pi}{N}nk}, k = 0, \dots, N-1.$$

Definition

The inverse discrete Fourier transform (IDFT) of the complex sequence $\{X_n\}_{n=0}^{N-1}$, is the following sequence

$$x_k = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} X_n e^{i\frac{2\pi}{N}nk}, k = 0, \dots, N-1.$$

Discrete Fourier Transform



$$\cdot \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \end{pmatrix}$$

Discrete Fourier Transform

Proposition

If $\{x_n\}_{n=0}^{N-1}$ *is a sequence of real numbers and* $\{X_n\}_{n=0}^{N-1}$ *is its DFT, then*

$$X_{N-k} = \overline{X_k}, k = 1, \dots, N-1.$$

Discrete Fourier Transform

Proposition

If $\{x_n\}_{n=0}^{N-1}$ *is a sequence of real numbers and* $\{X_n\}_{n=0}^{N-1}$ *is its DFT, then*

$$X_{N-k} = \overline{X_k}, k = 1, \dots, N-1.$$

Plancherel's Theorem

If the DFT of the complex sequence $\{x_n\}_{n=0}^{N-1}$ is $\{X_n\}_{n=0}^{N-1}$, then

Discrete Fourier Transform

Proposition

If $\{x_n\}_{n=0}^{N-1}$ *is a sequence of real numbers and* $\{X_n\}_{n=0}^{N-1}$ *is its DFT, then*

$$X_{N-k} = \overline{X_k}, k = 1, \dots, N-1.$$

Plancherel's Theorem

If the DFT of the complex sequence $\{x_n\}_{n=0}^{N-1}$ is $\{X_n\}_{n=0}^{N-1}$, then

$$\sum_{n=0}^{N-1} |x_n|^2 = \sum_{n=0}^{N-1} |X_k|^2$$

Convolution

Definition

Convolution of the sequences of real numbers $x = \{x_n\}_{n=-\infty}^{+\infty}$, $y = \{y_n\}_{n=-\infty}^{+\infty}$ *is the following sequence:*

$$(x * y)_n =$$

Convolution

Definition

Convolution of the sequences of real numbers $x = \{x_n\}_{n=-\infty}^{+\infty}$, $y = \{y_n\}_{n=-\infty}^{+\infty}$ *is the following sequence:*

$$(x * y)_n = \sum_{k=-\infty}^{+\infty} x_k y_{n-k}.$$

Convolution

Definition

Convolution of the sequences of real numbers $x = \{x_n\}_{n=-\infty}^{+\infty}$, $y = \{y_n\}_{n=-\infty}^{+\infty}$ *is the following sequence:*

$$(x * y)_n = \sum_{k=-\infty}^{+\infty} x_k y_{n-k}.$$

Let $x = \{x_n\}_{n=0}^{N-1}$, $y = \{y_n\}_{n=0}^{N-1}$ are finite sequences of real numbers.

Convolution

Definition

Convolution of the sequences of real numbers $x = \{x_n\}_{n=-\infty}^{+\infty}$, $y = \{y_n\}_{n=-\infty}^{+\infty}$ *is the following sequence:*

$$(x * y)_n = \sum_{k=-\infty}^{+\infty} x_k y_{n-k}.$$

Let $x = \{x_n\}_{n=0}^{N-1}$, $y = \{y_n\}_{n=0}^{N-1}$ are finite sequences of real numbers. Denote

$$z_n = \sum_{k=0}^{N-1} x_k y_{(n-k)_N}.$$

Convolution

Definition

Convolution of the sequences of real numbers $x = \{x_n\}_{n=-\infty}^{+\infty}$, $y = \{y_n\}_{n=-\infty}^{+\infty}$ *is the following sequence:*

$$(x * y)_n = \sum_{k=-\infty}^{+\infty} x_k y_{n-k}.$$

Let $x = \{x_n\}_{n=0}^{N-1}$, $y = \{y_n\}_{n=0}^{N-1}$ are finite sequences of real numbers. Denote

$$z_n = \sum_{k=0}^{N-1} x_k y_{(n-k)_N}.$$

This operation is called circular convolution. The convolution theorem is true for circular convolution.

Spectrum and Phase

Definition

Let $x = \{x_n\}_{n=0}^{N-1}$ *be a sequence of real numbers and* $\{X_n\}_{n=0}^{N-1}$ *is its DFT. The spectrum of* x *is the sequence* $\{|X_n|\}_{n=0}^{N-1}$.

Spectrum and Phase

Definition

Let $x = \{x_n\}_{n=0}^{N-1}$ *be a sequence of real numbers and* $\{X_n\}_{n=0}^{N-1}$ *is its DFT. The spectrum of* x *is the sequence* $\{|X_n|\}_{n=0}^{N-1}$.

Definition

Let $x = \{x_n\}_{n=0}^{N-1}$ *be a sequence of real numbers and* $\{X_n\}_{n=0}^{N-1}$ *is its DFT. The phases of* x *is the sequence* $\{\arg X_n\}_{n=0}^{N-1}$.